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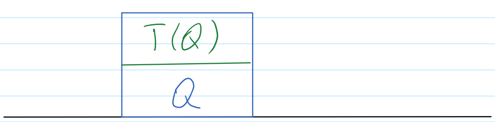
Theorem (Garban–Rohde–Schramm)
Let Ω be a simply connected domain, $a, b \in \widehat{\partial\Omega}$ – prime ends. Let γ be SLE_κ in (Ω, a, b).
Then $\gamma[0,\infty)$ is a.s. a curve for $\kappa < 8$.

The case $\kappa \le 4$ is trivial: an image of a simple curve under conformal map is a simple curve.

What we are afraid of:

Proof. Let us first show that outside of a small set every conformal map is Hölder. Take $p = 8/\kappa - 1$ (what we discuss works for any $0 < p < 1$).



Let $\mathcal{M}_n = \{ Q = \{ 0 \le y \le 2^{-n}, \ k2^{-n} \le x \le (k+1)2^{-n} \} \}$ – collection of squares with side $2^{-n} =: \ell(Q)$
Let $T(Q) := \{ 2^{-n-1} \le y \le 2^{-n}, \ k2^{-n} \le x \le (k+1)2^{-n} \}$ for $Q \in \mathcal{M}_n$.

Let γ be SLE_κ in $(\mathbb{H}, 0, \infty)$. By the estimate (α): Dimension of SLE curves., we have that $\forall Q$: $P(\gamma \cap Q \ne \emptyset) \asymp \ell(Q)^p$.

Let $\varphi : \mathbb{H} \to \Omega$ be conformal, Ω – bounded, $\underbrace{\hat\varphi(0) = a, \hat\varphi(\infty) = b}$ prime ends
Let $\mathcal{L}_N := \{ Q \in \mathcal{M}_n, n \ge N : \int_{T(Q)} |\varphi'|^2 > 2^{-np} \}$
Note that $\sum \ell(Q)^p \le \sum \int |\varphi'|^2 \le \text{Area}\{ \varphi(z) : \operatorname{Im} z \le 2^{-N} \}$

Let $\mathcal{L}_N := \{ Q \in \mathcal{D}_n, n \geq N : \int_{T(Q)} |\varphi'|^2 > \ell(Q)^p \}$

Note that $\sum_{Q \in \mathcal{L}_N} \ell(Q)^p \leq \sum_{Q \in \mathcal{L}_N} \int_{T(Q)} |\varphi'|^2 \leq \text{Area}\{\varphi(z) : \operatorname{Im} z \leq 2^{-N}\}$

So $\forall \varepsilon > 0 \ \exists N : \quad P(\gamma \cap \mathcal{L}_N \neq \emptyset) < \varepsilon$

Let us now fix $R < \infty$, $T_R = \inf\{t : |\gamma(t)| \geq R\}$

Observe: Hyperbolic diameter of $T(Q) \asymp 1$.

So if $z \notin \bigcup_{Q \in \mathcal{L}_N} Q$, $|z| \leq R$, then either $\operatorname{Im} z \geq 2^{-N}$ (and $|\varphi'(z)| \lesssim 1$).

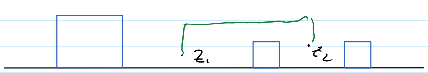
$z \in T(Q)$ for $Q \notin \mathcal{L}_N \Rightarrow \int_{T(Q)} |\varphi'|^2 \leq \ell(Q)^p \Rightarrow$ Bloch, $\forall z_1, z_2 \in T(Q)$ $\left| \frac{\varphi'(z_1)}{\varphi'(z_2)} \right| \lesssim 1$

$|\varphi'(z)|^2 \ell(Q)^2 \lesssim \ell(Q)^p \Rightarrow$

$|\varphi'(z)| \lesssim \ell(Q)^{1-\frac{p}{2}} \asymp (\operatorname{Im} z)^{1-\frac{p}{2}}$.

So if $|z_1|, |z_2| \leq R$; $z_1, z_2 \notin \bigcup_{Q \in \mathcal{L}_N} Q$, then

$|\varphi(z_1) - \varphi(z_2)| \lesssim |z_1 - z_2|^{p/2}$ — first check for $z_1, z_2 \in T(Q)$ - same Q, $\operatorname{Re} z_1 = \operatorname{Re} z_2$ (by integration) then join arbitrary z_1, z_2 by a path outside of $\bigcup_{Q \in \mathcal{L}_N} Q$.



So if $\gamma(0, T_R) \cap \bigcup_{Q \in \mathcal{L}(Q)} Q = \emptyset$, then $\varphi(\gamma(0, T_R))$ is a curve, as an image of a curve under Hölder map.

Then $P(\varphi(\gamma(0, T_R)) \text{ is a curve}) > 1 - \varepsilon$.

Now let $\varepsilon \to 0$, $R \to \infty$ to get that

$P(\varphi(\gamma(0, \infty)) \text{ is a curve}) = 1$

SLE$_\kappa$ from a to b.

For unbounded Ω, do everything in spherical metric.

Reversibility of SLE.

Ω - simply connected domain, $a, b \in \partial\hat{\Omega}$

Is the law of SLE_κ from a to b is the same as from b to a?

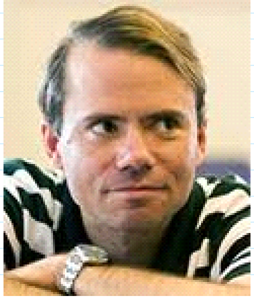
No, if $\kappa \geq 8$ - Rohde-Schramm.

Yes, $\kappa \leq 4$ - Dapeng Zhan.

Dapeng Zhan

Yes, $4 < \kappa < 8$ Jason Miller, Scott Sheffield.

Jason Miller



Scott Sheffield